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FREQUENCY BASED DESIGN OF GENERALIZED MINIMUM VARIANCE CONTROL WITH QUASI-SLIDING MODE

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Abstract: *In this paper, a new approach to design of generalized minimum variance control with quasi-sliding mode, based on the well known frequency design methods, is discussed. The structure of proposed controller is considered first, and, then, it is briefly presented how to choose the controller parameters by using Bode plots approach. The overall procedure is verified by digital simulation and experimental results. We expect that the given design procedure will enable servo engineers better insight and understanding of sliding mode control systems.*

Key Words: ***Discrete-time Sliding Mode Control/Bode plots/ Phase Lead Compensator***

1. INTRODUCTION

The modern implementation of sliding mode control, based on microcontrollers, assumes the discretization process of sliding mode control, causing the quasi-sliding motion and undesirable chattering phenomenon in the $O(T)$ vicinity of a sliding hypersurface. Generally speaking, the digital sliding mode control design methods could be divided into two groups: the techniques based on the state-space model representations, wherein the use of observer is mandatory, and the methods utilizing only the output measurements.

The input/output based digital sliding mode controls are mainly established on the use of generalized minimum variance (GMV) control concept [1, 2]. This approach represents the combination of GMV and sliding mode controls where one component recruits the another one in such manner that the former control enables the input/output control approach, while the later provides the system robustness to parameter perturbation and external disturbances.

In order to obtain zero tracking error, the traditional GMV control [3] requires reference input signal known in advance and the absence of external disturbance (the fully compensated system). With tracking systems, reference input is never known in advance; and, moreover, there are many systems with no information on reference input signal at all, but with only knowledge of the error signal (for instance: optical disc drive tracking loop [4]). In these cases, there are always some varying parameters, and the elimination of external disturbances is hardly possible. One solution to this problem is the combination of the GMV control and the relay control component which is filtered through the digital integrator reducing undesired chattering in that way [1, 5]. The further alleviation of chattering is done by using fuzzy sliding mode control concept [6, 7].

The main design question, how to choose the controller parameters to ensure the desired system motion in quasi-sliding mode, is mainly treated in two ways. One approach is discussed in [1, 8] and it is developed on the similarity between the digital equivalent control and the GMV control. Unfortunately, this solution is rather time consuming. Moreover, the design problem becomes more difficult in the case of parameter perturbations [1].

In order to make the design procedure more easer and comprehensive for the servo engineers with the little knowledge of sliding mode control systems, we propose the design of the GMV control with quasi-sliding mode based on frequency design principles using Bode plots of system. The choice of quasi-sliding mode controller parameters is straightforward even in the case of significant variations of plant parameters.

The paper is organized as follows. The problem formulation is given in the section 2. The GMV control is presented in section 3, while the GMV control with quasi-sliding mode is discussed in section 4. Section 5 briefly elaborates the closed loop system dynamics. The procedure of parameter selection is explained in section 6. The paper concludes with illustrative example in section 7, showing both simulation and experimental results of new approach in design of servo system.

2. PROBLEM FORMULATION

Let us consider a single-input-single-output plant with the following state space model:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) + \mathbf{d}f(t), \\ y(t) &= \mathbf{c}\mathbf{x}(t), \end{aligned} \tag{1}$$

where: $\mathbf{x}(t) = [x_1(t) \quad x_2(t) \quad \ldots \quad x_n(t)]^{\mathrm{T}} \in R^n$ represents a state space vector, $u(t) \in R$ a plant input, $f(t) \in R$ a external disturbance, $y(t) \in R$ a plant output, and n a order of plant. A matrix $\mathbf{A}$ and vectors $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ have the following dimensions: $\mathbf{A} = [a_{ij}]_{n\times n}$, $\mathbf{b} = [b_i]_{n\times 1}$, $\mathbf{c} = [c_j]_{1\times n}$, $\mathbf{d} = [d_i]_{n\times 1}$.

Suppose that the reference input signal is not available, but the error signal is measurable, and assume that the pair (**A**, **c**) is observable i.e. the plant model (1) is given in observable canonical form. The substitution of $e(t) = y(t) - r(t)$ in (1), where $r(t)$ is the reference input signal, gives the plant model in the error signal state-space as:

$$\begin{aligned}
&\dot{\mathbf{e}}(t) = \mathbf{A}\mathbf{e}(t) + \mathbf{b}u(t) + \mathbf{d}f(t) + \mathbf{g}(t),\\
&\mathbf{g}(t) = \mathbf{p}_1\dot{r}(t) + \mathbf{p}_0 r(t),\\
&\mathbf{p}_0 = \begin{bmatrix} a_{n-1} & a_{n-2} & \dots & a_1 & a_0 \end{bmatrix}^{\mathrm{T}},\ \mathbf{p}_0 = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix}^{\mathrm{T}},\\
&y = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}\mathbf{e}(t),\ \mathbf{e}(t) = \begin{bmatrix} e(t) & x_2(t) & \dots & x_n(t) \end{bmatrix}^{\mathrm{T}}.
\end{aligned} \tag{2}$$

After the implementation of the ideal sampler and the zero order hold circuit, the discrete-time state-space plant model can be represented in the form of:

$$\begin{aligned}
&e_{k+1} = \mathbf{\Phi} e_k + \boldsymbol{\gamma} u_k + \mathbf{h}_k + \boldsymbol{\eta}_k,\\
&y_k = e_k,
\end{aligned} \tag{3}$$

$$\mathbf{\Phi} = e^{\mathbf{A}T},\ \boldsymbol{\gamma} = \int_0^T e^{\mathbf{A}\lambda}\mathbf{b}\,\mathrm{d}\lambda, \tag{4}$$

$$\mathbf{h}_k = \int_0^T e^{\mathbf{A}\lambda}\mathbf{d}f((k+1)T - \lambda)\,\mathrm{d}\lambda, \tag{5}$$

$$\boldsymbol{\eta}_k = \int_0^T e^{\mathbf{A}\lambda}\mathbf{g}((k+1)T - \lambda)\,\mathrm{d}\lambda, \tag{6}$$

$\mathbf{e}_k = \begin{bmatrix} e_{1k} & e_{2k} & \dots & e_{nk} \end{bmatrix}^{\mathrm{T}}$, T is a sampling period. Note that $\bullet_k = \bullet(kT)$. It is obvious that the assumptions $|f(t)| < \mathrm{F}$ and $|\mathbf{g}(t)| < \mathbf{G}$, where $F < \infty$ and $\mathbf{G} < \infty$, imply $\mathbf{h}_k = O(T)$ and $\boldsymbol{\eta}_k = O(T)$.

The model of the system in the z-domain is then obtained as:

$$e_k = \frac{z^{-1}B(z^{-1})}{A(z^{-1})}u_k + \frac{z^{-1}\mathbf{D}(z^{-1})}{A(z^{-1})}(\mathbf{h}_k + \boldsymbol{\eta}_k), \tag{7}$$

where z^{-1} is the unit delay, $A(z^{-1}) = z^{-n}\det(z\mathbf{I} - \mathbf{\Phi})$, $B(z^{-1}) = z^{-n+1}\mathbf{c}[\mathrm{adj}(z\mathbf{I} - \mathbf{\Phi})\boldsymbol{\gamma}]$, $\mathbf{D}(z^{-1}) = z^{-n+1}\mathbf{c}[\mathrm{adj}(z\mathbf{I} - \mathbf{\Phi})]$.

The goal of design is to ensure quasi-sliding mode in the vicinity of sliding hypersurface $s_k = 0$, where:

$$s_{k+1} = C(z^{-1})(y_{k+1} - r_{k+1}) + Q(z^{-1})u_k, \tag{8}$$

represents a switching function of a GMV control with quasi-sliding mode control. Thereby, $C(z^{-1})$ is Jury's polynomial and $Q(z^{-1})$ is the polynomial satisfying $Q(1) = 0$.

In steady-state, when $k \to \infty$ i.e. $z \to 1$, the plant output is defined as:

$$y_\infty = r_\infty + \frac{s_\infty}{C(1)}, \tag{9}$$

so the system steady-state accuracy depends on the accuracy of switching function s_k in such manner that the smaller s_k implies the smaller error signal e_k.

In this paper we consider the new approach in design of the GMV control with quasi-sliding mode using Bode plots for the tracking system where the reference input signal is not known at all, i.e. we can measure only the error signal $e_k = y_k - r_k$ at k and the previous time instants.

3. GMV CONTROL

The GMV control in the case with unknown reference input signal is given in the following form:

$$u_k = -\frac{F(z^{-1})e_k}{E(z^{-1})B(z^{-1}) + Q(z^{-1})}, \tag{10}$$

where the polynomials $E(z^{-1})$ and $F(z^{-1})$ are the solutions of the so-called Diophantine equation:

$$E(z^{-1})A(z^{-1}) + z^{-1}F(z^{-1}) = C(z^{-1}), \tag{11}$$

The implementation of (10) in (7), taking into account (8) and (11), yields:

$$s_{k+1} = E(z^{-1})\mathbf{D}(z^{-1})(\mathbf{h}_k + \boldsymbol{\eta}_k) = O(T). \tag{12}$$

Since the error signal is determined by $s_\infty / C(1)$ in the steady-state, the plant output y_k will track the reference input signal r_k in $O(T)/C(1)$ boundaries in that case.

4. GMV CONTROL WITH QUASI-SLIDING MODE

The better system steady-state accuracy can be reached if the quasi-sliding mode control is introduced into the GMV control. The result is a control in the form of:

$$u_k = -\frac{F(z^{-1})e_k + \dfrac{\alpha T}{1 - z^{-1}}\mathrm{sgn}(s_k)}{E(z^{-1})B(z^{-1}) + Q(z^{-1})}. \tag{13}$$

Notice that (13) differs from the GMV control (10) in the relay control component which is filtered through the discrete-time integrator. This approach leads to significant chattering reduction in quasi-sliding mode. The substitution of (13) in (7), by using (8) and (11), gives:

$$\begin{aligned}
s_{k+1} = {} & s_k - \alpha T\,\mathrm{sgn}(s_k) + \\
& + E(z^{-1})\mathbf{D}(z^{-1})(\mathbf{h}_k - \mathbf{h}_{k-1} + \boldsymbol{\eta}_k - \boldsymbol{\eta}_{k-1}),
\end{aligned} \tag{14}$$

According to (5) and (6), the differences $\mathbf{h}_k - \mathbf{h}_{k-1}$ and $\boldsymbol{\eta}_k - \boldsymbol{\eta}_{k-1}$ can be expressed as:

$$\mathbf{h}_k - \mathbf{h}_{k-1} = \int_0^T e^{\mathbf{A}\lambda}\mathbf{d}\int_{kT-\lambda}^{(k+1)T-\lambda}\dot{f}(\rho)\,\mathrm{d}\rho\,\mathrm{d}\lambda, \tag{15}$$

$$\boldsymbol{\eta}_k - \boldsymbol{\eta}_{k-1} = \int_0^T e^{\mathbf{A}\lambda}\int_{kT-\lambda}^{(k+1)T-\lambda}\dot{\mathbf{g}}(\rho)\,\mathrm{d}\rho\,\mathrm{d}\lambda, \tag{16}$$

implying that they posses $O(T^2)$ precision if $\left|\dot{f}(t)\right| < F' = \text{const.}$ and $\left|\dot{\mathbf{g}}(t)\right| < \mathbf{G}' = \text{const.}$ i.e.:

$$\mathbf{h}_k - \mathbf{h}_{k-1} = O(T^2), \tag{17}$$

$$\boldsymbol{\eta}_k - \boldsymbol{\eta}_{k-1} = O(T^2), \tag{18}$$

and consequently:

$$s_{k+1} = O(T^2). \tag{19}$$

In other words, the GMV control with quasi-sliding mode provides the zero error signal for step reference input signal and/or constant external disturbance, while in the case with ramp reference input signal and/or external disturbance this error will be in $O(T^2)/C(1)$ boundaries.

The controller parameter α is selected in accordance with the following *Lemma*.

Lemma 1: *Consider the discrete-time control system (7), (13), with switching function (8), whose dynamics is defined by (14). If parameter α is chosen to satisfy the follwing inequality:*

$$\alpha T > \max\left|E(z^{-1})\mathbf{D}(z^{-1})(\mathbf{h}_k - \mathbf{h}_{k-1} + \boldsymbol{\eta}_k - \boldsymbol{\eta}_{k-1})\right| = O(T^2), \tag{20}$$

then there exists a possitive integer number $K_0 = K_0(s_0)$, so that quasi-sliding mode is established in the domain $S(T^2)$ determined by:

$$S(T^2) = \left\{\begin{aligned} & s_k : \left|s_k\right| < \alpha T + \\ & + \max\left|E(z^{-1})\mathbf{D}(z^{-1})(\mathbf{h}_k - \mathbf{h}_{k-1} + \boldsymbol{\eta}_k - \boldsymbol{\eta}_{k-1})\right| = \\ & = O(T^2) \end{aligned}\right\} \tag{21}$$

for each $k \geq K_0$ where $\alpha = O(T)$.

The proof of the *Lemma 1* is given in [1].

5. CLOSED-LOOP SYSTEM DYNAMICS

On the basis of (7) and (8), the closed-loop system dynamics can be described by:

$$\begin{aligned} e_k = & \frac{B(z^{-1})}{B(z^{-1})C(z^{-1}) + A(z^{-1})Q(z^{-1})} s_k + \\ & + \frac{Q(z^{-1})\mathbf{D}(z^{-1})}{B(z^{-1})C(z^{-1}) + A(z^{-1})Q(z^{-1})} (\mathbf{h}_{k-1} + \boldsymbol{\eta}_{k-1}). \end{aligned} \tag{22}$$

The overall system stability is defined by the following *Lemma*.

Lemma 1: *System described by (7) and (13) is stable iff:*

1) *the inequality (20) is valid for each k, i.e. the quasi-sliding mode exists in the system, and*
2) *the polynomial:*

$$B(z^{-1})C(z^{-1}) + A(z^{-1})Q(z^{-1}) \tag{23}$$

has all its roots inside the unit disc in the z-plane, and the pairs $(B(z^{-1}), Q(z^{-1}))$, $(C(z^{-1}), A(z^{-1}))$ and $(C(z^{-1}), Q(z^{-1}))$ have no common roots outside this unit disc.

6. SELECTION PROCEDURE OF CONTROLLER PARAMETERS

The main design question is how to choose the polynomials $C(z^{-1})$ and $Q(z^{-1})$ to ensure the desired system motion in quasi-sliding mode. As we have already told, one approach is discussed in [1, 8] and it is developed on the similarity between the digital equivalent control and the GMV control. In this paper, we suggest a different design procedure based on the use of Bode plots. The idea is to use the frequency design methods in order to determine the polynomials $F(z^{-1})$ and

$$G(z^{-1}) = E(z^{-1})B(z^{-1}) + Q(z^{-1}). \tag{24}$$

The polynomials $C(z^{-1})$ and $Q(z^{-1})$ is calculated then directly from (11) and (24). In other words, we obtain herein the polynomials $F(z^{-1})$ and $G(z^{-1})$ as a result of phase lead compensator design using Bode plots [9].

The purpose of phase lead compensator design in the frequency domain generally is to satisfy specifications on phase margin and on gain crossover frequency or closed-loop bandwidth. A phase margin specification can represent a requirement on relative stability due to pure time delay in the system, or it can represent desired transient response characteristics that have been translated from the time domain into the frequency domain. The overall philosophy is for the compensator to adjust the system's Bode phase curve to establish the required phase margin at the existing gain-crossover frequency, ideally without disturbing the system's magnitude curve at that frequency and without reducing the zero-frequency magnitude value. The unavoidable shift in the gain crossover frequency is a function of the amount of phase shift that must be added to satisfy the phase margin requirement.

The transfer function of phase lead compensator can be presented in the form of:

$$G_c^{lead}(s) = \frac{F_c(s)}{G_c(s)} = \frac{K_c(s + z_c)}{\beta(s + p_c)} = K_c \frac{\tau s + 1}{\beta \tau s + 1} \tag{25}$$

where

$$z_c > 0,\ p_c > 0,\ \beta = \frac{z_c}{p_c} < 1,\ \tau = \frac{1}{z_c} = \frac{1}{\beta p_c}. \tag{26}$$

The following steps outline the procedure that will be used to design the phase lead compensator to satisfy phase margin specifications:

1) determine the amount of phase shift in the plant frequency transfer function $W(j\omega)$ at the gain crossover frequency and calculate the uncompensated phase margin;
2) calculate the values for $\phi_{\max}$ and β that are required to raise the phase curve to the value needed to satisfy the phase margin specification;
3) determine the value for the final gain crossover frequency;
4) using the value of β and the final gain crossover frequency, compute the lead compensator's zero z_c and pole p_c;

5) using bilinear transformation calculate $G_d^{lead}(z^{-1}) = F(z^{-1})/G(z^{-1})$ from $G_c^{lead}(s)$.

7. ILLUSTRATIVE EXAMPLE

The controller design procedure, elaborated above, is verified on the experimental setup consists of *Inteco* modular servo system presented in Fig. 1.

Fig. 1 *Modular servo system*

The transfer function of servo system is given by:

$$W(s) = \frac{180}{s(0.8607s+1)} \qquad (27)$$

We assume that there is no external disturbances acting on the system. After the implementation of the ideal sampler and the zero-order hold circuit, the discrete-time plant model becomes:

$$e_k = \frac{z^{-1}B(z^{-1})}{A(z^{-1})}u_k,$$
$$A(z^{-1}) = 1 - 1.9884z^{-1} + 0.9884z^{-2}, \qquad (28)$$
$$B(z^{-1}) = 0.0104 + 0.01036z^{-1},$$

with the sampling period $T = 0.01\text{s}$. We specify the desired system phase margin to be 60° and applying frequency based design procedure the transfer function of phase lead compensator is:

$$G_c^{lead}(s) = \frac{21.29s + 144.6}{s + 144.6}, \qquad (29)$$

in the continious-time domain, while its discrete-time version is given by:

$$G_d^{lead}(z^{-1}) = \frac{12.78 - 11.94z^{-1}}{1 - 0.1609z^{-1}} = \frac{F(z^{-1})}{G(z^{-1})}. \qquad (30)$$

Figs. 2 and 3 show the continuous- and discrete-time Bode plots for phase lead compensated systems, respectively.

The polynomials $C(z^{-1})$ and $Q(z^{-1})$ are computed by using (11) and (24) in the following form:

$$C(z^{-1}) = 53.1347 - 92.1873z^{-1} + 39.8917z^{-2},$$
$$Q(z^{-1}) = 0.5796(1 - z^{-1}). \qquad (31)$$

The controller parameter α is chosen to be equal to 1.

The reference input signal, not known but indirectly contained in error signal, is determined by:

$$r(t) = \begin{cases} 50t & 0 \le t < 4 \\ -12{,}5((t-4)^2 - 16) & 4 \le t < 8 \\ -50(t-8) & 8 \le t < 12 \\ 12{,}5((t-12)^2 - 16) & 8 \le t < 16 \end{cases} \qquad (32)$$

and shown in Fig. 4. Since it is not available we consider it as external disturbance according to (3) and (7).

Figs. 5 and 6 present the simulation and experimental results of the error signal response for the GMV control, respectively. According to (12), the swithing function s_k and, consequently, the error signal should have ramp i.e. parabolic shape taking into account the form of the reference input signal (32). This is not the case with both simulation and experimental results since if the system type is 1, then the polynomial $A(z^{-1})$ contains the first difference, causing the system response as it is shown in Figs. 5 and 6 [1].

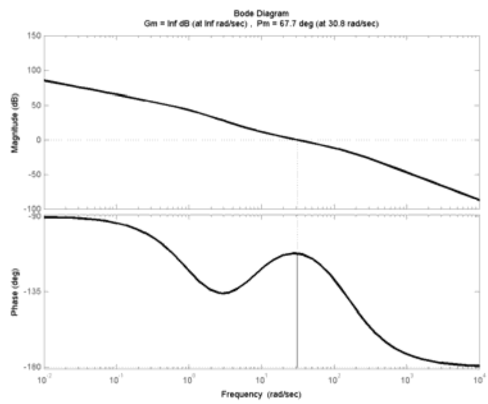

Fig. 2 *Continuous-time Bode plots for lead compensated system*

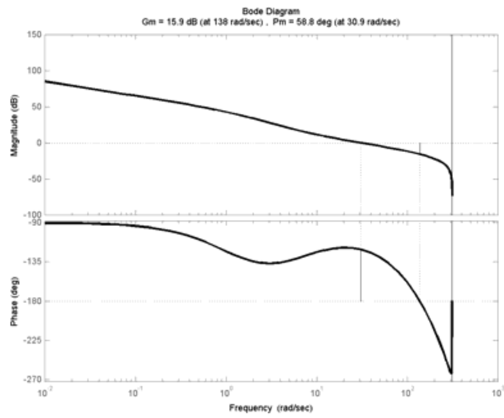

Fig. 3 *Discrete-time Bode plots for lead compensated system*

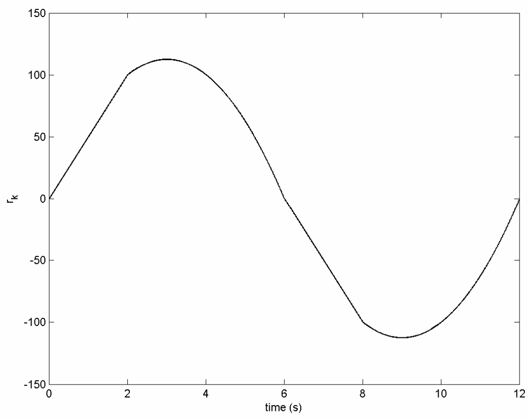

Fig. 4 *Reference input signal*

The simulation and experimental results of the error signal response for the GMV control with quasi-sliding mode is given in Figs 7 and 8. The latter control yields better results in comparison with the GMV control, what is obvious since the GMV control with quasi-sliding mode has the better system steady-state accuracy. In both cases, the simulation and experimental results are rather similar.

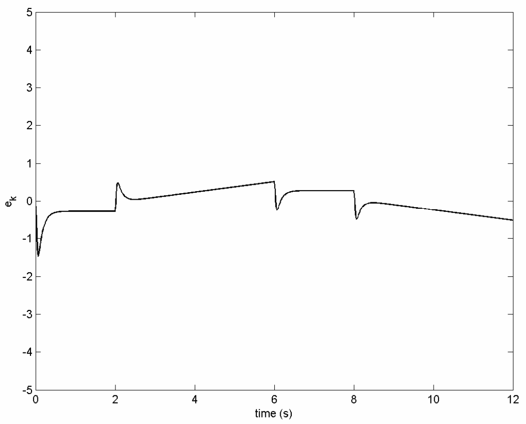

Fig. 5 *System response: GMV control (simulation results)*

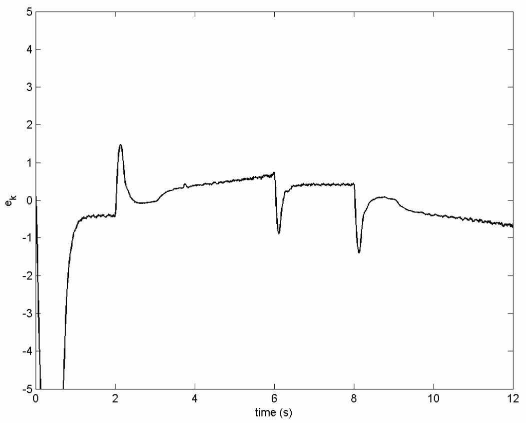

Fig. 6 *System response: GMV control (experimental results)*

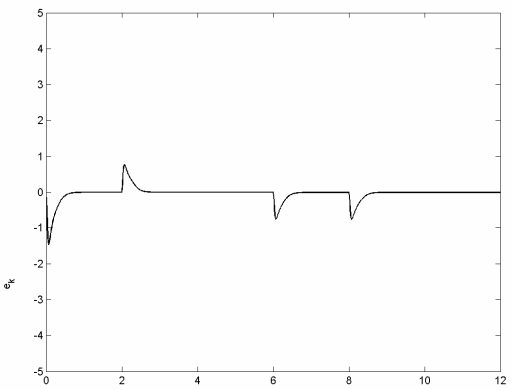

Fig. 7 *System response: GMV controlwith quasi-sliding mode (simulation results)*

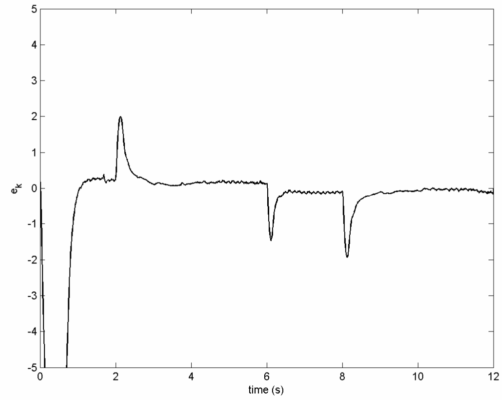

Fig. 8 *System response: GMV controlwith quasi-sliding mode (experimental results)*

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